Effect of boundary conditions on scaling in the ''game of Life''

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The debate as to whether the ''game of Life'' is self-organized critical remains unresolved. We present evidence that boundary conditions play an important role in the scaling behavior, resulting in apparently contradictory results. We develop an analytic form for the scaling function, and demonstrate that periodic boundaries force saturation, while open boundaries exhibit no such transitions on similar scales. We also consider the removal of boundaries altogether. [S1063-651X(97)10704-8]

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Some of the most interesting and complex dynamics have been observed in systems which exhibit self-organized criticality (SOC) [2], a description of systems which naturally tend toward a critical state (lacking any natural length scales) without requiring any tuning of external parameters. For the last several years there has been a debate as to whether the ''game of Life'' (GL) [1], a two-dimensional cellular automaton, exhibits self-organized criticality. In the case of the GL it is suggested this takes the form of a power-law distribution of relaxation times after the system is perturbed from equilibrium, $D(t) \propto t^{-b}$, where $b \approx 1.4$.

The ''game of Life'' is defined on a square two-dimensional lattice of sites which are either *alive* or *dead*. The lattice is updated synchronously, and the rules governing the evolution vaguely mimic logistic dynamics: (1) A live site will remain alive if exactly two or three of its eight nearest-neighbors are alive; otherwise it will die. (2) If a dead site has exactly three live neighbors, it will be toggled to the live state (birth).

Much of the debate [3–6] has centered around finite-size computer simulations of the GL for which differing bound-ary conditions have been applied. On small lattices the power-law description seems justified, but it is uncertain whether this extends to the thermodynamic limit. Near a critical point the correlation length diverges, so the boundary conditions may not play a diminishing role as the lattice grows. To determine how the dynamics scale to the thermodynamic limit we apply a finite-scaling technique and make an explicit fit of the scaling function.

We considered both periodic and cold (often called ''open'') boundary conditions on square lattices of side L. Cold boundaries assume that all cells beyond the boundaries are *dead*, and periodic boundaries associate each point on an edge with a corresponding point on the far edge. The initial equilibrium state was generated from an initially random 50% occupancy, and 10 000 random site perturbations (flipping either a dead site to live, or vice versa) were performed, each after the previous avalanche had stabilized. We found no evidence of perturbation-site dependencies or initial transients. To avoid ''flip fails'' (perturbations originating in dead regions which decay to the original state in just one time step) we restricted candidate perturbation sites to just
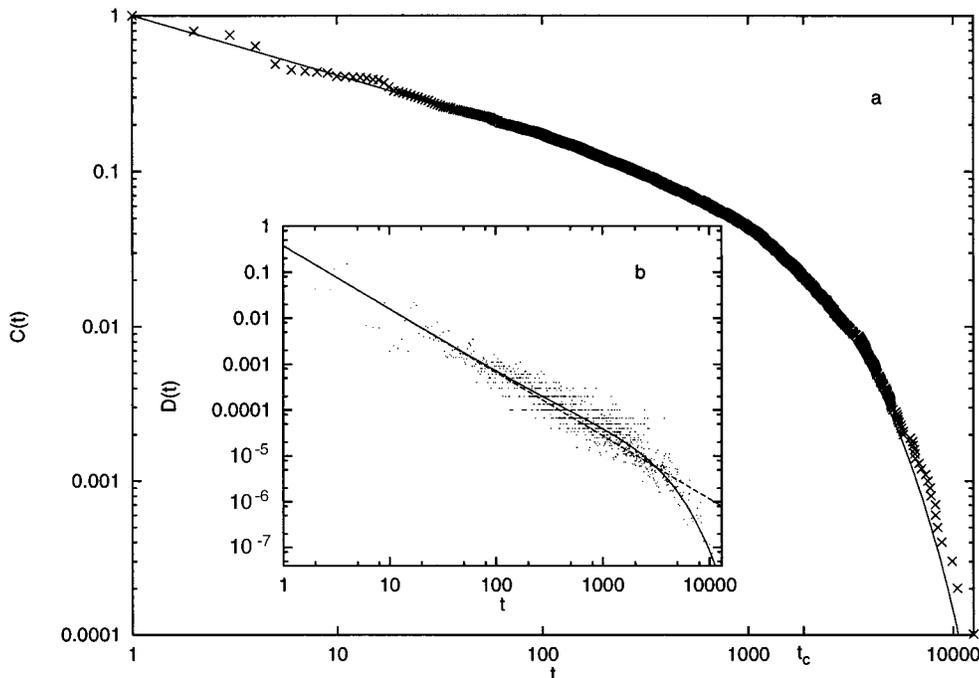
FIG. 1. Normalized distributions of lifetimes t of avalanches produced by repeated single-site perturbations in the ''game of Life'' on a 256×256 lattice with periodic boundaries. The cumulative distribution (a) is fitted to a decaying power law with critical exponent $b=1.376$ and critical lifetime $t_c=1865$. (b) The derivative of this fit (solid line) is compared to the original distribution and a power law neglecting drop-off (dashed line).
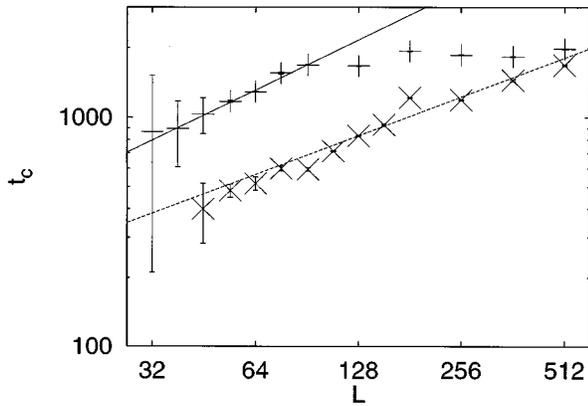
FIG. 2. Scaling behavior of the critical lifetime in the ''game of Life'' for periodic ($+$) and cold ($\times$) boundaries. The error bars are calculated assuming no correlations in the lifetimes of avalanches. Also shown are power-law fits for both periodic (solid line) and cold (dashed line) boundaries with exponents $z_p = 0.72 \pm 0.05$ and $z_c = 0.56 \pm 0.06$, respectively. Notice the transition away from finite-scaling for large, periodic lattices.

FIG. 3. Scaling behavior of the average decay time of avalanches in the ''game of Life'' for periodic ($+$) and cold ($\times$) boundaries. Assuming finite scaling for the critical lifetime, we can predict a power law for both periodic (solid line) and cold (dashed line) boundaries with exponents $(2-b)z$ which evaluate to 0.45 ± 0.03 and 0.35 ± 0.04, respectively. Again, we observe a transition away from finite scaling for periodic lattices on scales larger than $L \approx 128$.

live sites and their nearest neighbors. The temporal periodicity of the lattice, indicating that the avalanche has stabilized, was tested statistically by the *activity* of the system (the sum of all flips in each time step, be it dead to alive, or vice versa), the requirement being that the activity must cycle with some period $p \leqslant 12$ for 24 time steps. This method was chosen to facilitate the removal of boundaries altogether. While it is possible that stable, mobile structures such as *gliders* could give the illusion of stability under this scheme, it seldom happened in practice, and did not significantly affect the dynamics.

By analyzing the cumulative distribution of avalanches, $C(t)$ (all avalanches larger than or equal to t), we were able to smooth the Gaussian noise observed in the original distribution $D(t)$, as shown in Fig. 1. We hypothesized that the cumulative distribution obeys a finite-scaling relationship of the form $C(t) = t^{1-b} g(t/t_c)$, where g is a scaling function and t_c depends on the lattice size. The scaling function has the following properties: (1) $g(x) \to 1$ for $x \ll 1$, and (2) $g(x) \to 0$ for $x \gg 1$.

We observed that the scaling function fit remarkably well to the form $g(t/t_c) = \exp(-t/t_c)$, where t_c is an adjustable fitting parameter, as is the critical exponent $1-b$. The point $t = t_c$ is called the critical lifetime, and represents a typical time scale up to which the power law is a valid description of the dynamics. Runs of various dimensions were fitted and the critical exponent agreed closely among them with an average for both boundaries and all length scales of $b = 1.377 \pm 0.009$, while the critical lifetime varied with L, as is shown in Fig. 2.

Given the above form for the cumulative distribution, the original distribution of lifetimes also obeys a finite-scaling law $D(t) = (b-1)t^{-b} f(t/t_c)$, where $f(x) = (1 + x/(b-1))\exp(-x)$. Notice that this scaling function has the peculiarity that it rises above unity near $t = (2-b)t_c$ [see Fig. 1(b)]. This anomalous ''hump'' may have misled some researchers' estimates of the critical exponent. The spread of data can be reproduced via a random

sampling of the distribution $D(t)$, suggesting that no correlations exist between successive avalanches.

Finite scaling predicts the transition t_c should scale with L as $t_c \propto L^z$. Figure 2 shows power-law fits for both periodic and cold boundaries with $z_p = 0.72 \pm 0.05$ and $z_c = 0.56 \pm 0.06$, respectively. The latter agrees favorably with Alstrøm and Leão's estimate of 0.52 [5]. The cold-boundary runs scale well with the lattice size, but there appears to be a transition away from finite-scaling for the periodic-boundary runs, around $L \approx 128$, to a constant $t_c = 1905 \pm 59$. Clearly, the GL with periodic boundaries does not exhibit SOC, but has both a characteristic size and time scale.

On scales that we have explored (up to $L = 512$) we see no evidence of saturation for cold boundaries. However, even for the largest lattice the critical lifetime ($t_c \approx 1685$) falls short of the average observed after crossover for periodic lattices. This suggests that we may need to explore larger lattices to observe a deviation from criticality. The estimated value of z_c predicts saturation for $L > 560$, but the large error margin in the proportionality constant may accomodate a lattice twice this size. Hence, to test whether the scaling behavior continues, and if the GL with cold boundaries is SOC, we may need to explore lattices on scales of $L > 1100$. The largest previous study using cold boundaries [5] tested up to $L = 1024$, with no observed transition away from finite scaling.

Some confusion has arisen from the study of the average decay time [4,5] instead of the critical lifetime as the statistic of choice for measuring scaling behavior. If the critical lifetime scales as proposed above, then we expect the average decay time to scale as $\langle t \rangle \propto L^{(2-b)z}$. From the above data we calculated $(2-b)z = 0.45 \pm 0.03$ and 0.35 ± 0.04 for periodic and cold boundaries, respectively. Calculating the proportionality constant from the integral $\langle t \rangle = \int_1^\infty t D(t) dt$, we compare the observed average decay time to the predicted scaling in Fig. 3. Periodic runs again exhibit saturation around
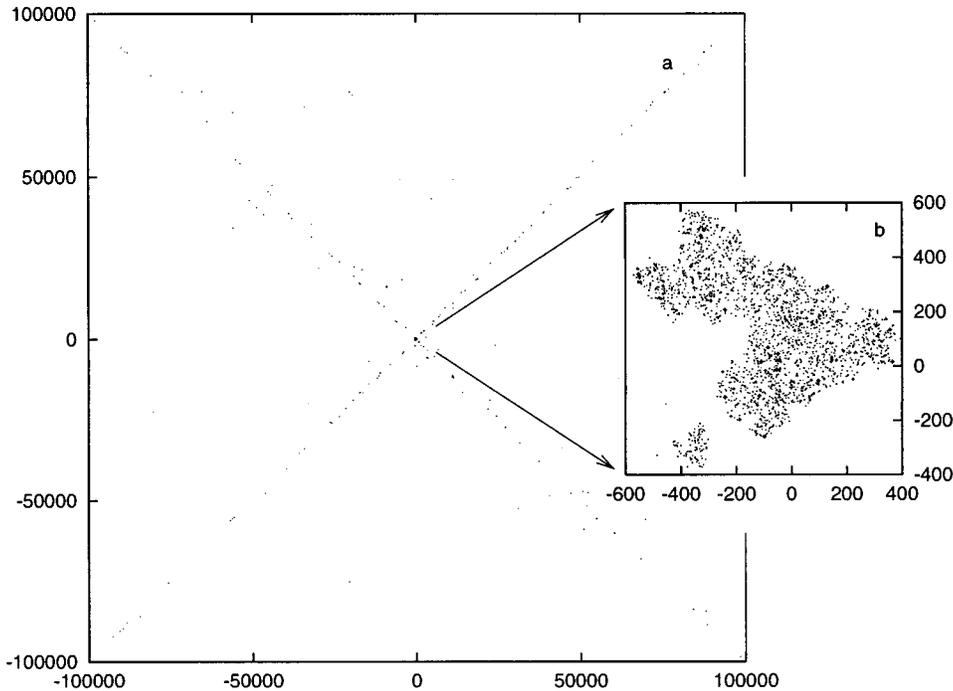
FIG. 4. Sample lattice configuration of the ''game of Life'' in the absence of boundaries after 1905 perturbations showing (a) two distinct regions, a relatively massive core of localized clusters, and a long trail of gliders radiating diagonally out to infinity. The largest cluster (b) has a characteristic size of $L \approx 1000$.

$L=128$, in agreement with Bennett and Bourzutschky [4], while cold runs fit closely on all scales, confirming Alstrøm and Leão's conclusions. The confusion is a direct result of using differing boundary conditions. Again, we stress that much larger lattices will be needed to resolve the issue as to whether cold boundaries produce SOC.

We also explored the effect of removing boundaries altogether, via dynamic memory allocation [7]. Initializing the space with an $L=50$ square lattice with 50% occupation at the origin, the system was allowed to develop naturally, with perturbations as before. Due to intensive computational demands, only 2000 perturbations were aqcuired. The final lattice configuration, shown in Fig. 4, reveals a sparse, fractal-like structure of live clusters which produce gliders (radiating diagonally outwards) while evolving. It appears that after an initial 1000 perturbation transient the dominant cluster (produced by the initial seed) ceased growing, and stabilized with a characteristic size of $L \approx 1000$. Using the same analytical technique as above, we estimate a critical lifetime $t_c = 1904 \pm 71$, further inclining us to believe that GL is subcritical with a characteristic avalanche lifetime. However, more data need to be aqcuired to determine if this behavior is significant or merely a by-product of the small initial seed.

The unbounded system is unusual in that gliders are never absorbed by boundaries, and radiate outwards (at a speed of one site diagonally per four time steps). Avalanches produce gliders fairly regularly, and the total count of live sites increases linearly with time (see Fig. 5). Unfortunately, the running time for a simulation then grows at least as the square of the number of perturbations.

The gliders may have another, even more important, effect. These gliders may be perturbed, producing stationary clusters in the ''glider path'' of other clusters. Eventually, the simulation will produce a large number of isolated clusters. Each perturbation will, with some probability, produce gliders. If any glider interacts with another cluster, it may

again produce other gliders, and so on. This hierarchy of avalanches produced by the long-range interactions of isolated clusters via gliders may produce dynamics entirely unique to the unbounded system. The tail of the lifetime distribution $D(t)$ may exhibit a Pareto-Lévy, or power-law, tail (see, for instance, [8]) which returns the behavior to the critical state.

In conclusion, our results indicate that boundaries play a crucial role in the scaling behavior of the ''game of Life,'' and the disparity in previous studies [3–6] is caused thereby. Periodic boundaries and the unbounded run suggest that GL follows a power law with exponent $b=1.377$ up to a critical lifetime, but then drops off exponentially. The critical lifetime scales with the lattice size only for small lattices, and seems to stabilize at $t_c = 1905$ in the thermodynamic limit, implying that the GL is subcritical. Independent research exploring the critical phase transition in a class of stochastic
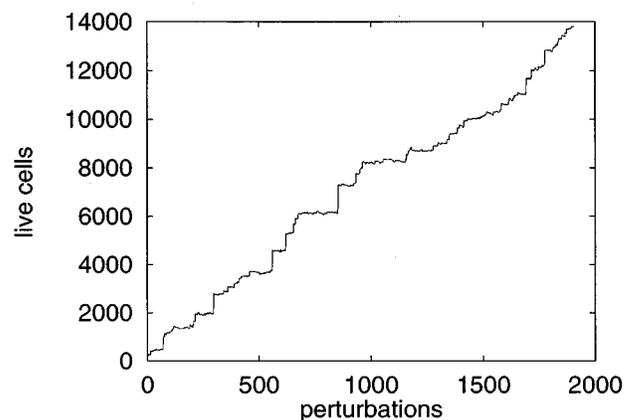
FIG. 5. In the absence of boundaries, the total number of live sites increases linearly with time (perturbations) in the ''game of Life.'' This is due mainly to the continual production of gliders by avalanches, which are no longer absorbed by boundaries.

cellular automata [9] seems to confirm this conclusion.

Cold boundaries appear to impede the scaling behavior such that the transition away from criticality cannot be seen on the scales we studied. We stress that, because the GL is at the least very close to criticality, further research is necessary to determine conclusively if the effects of cold boundaries play a diminishing role as the lattice grows. We also stress that the data for the unbounded lattice are extremely sparse, and must be supplemented before much credence can be given to them.

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